MCWP Senior High Mathematics League

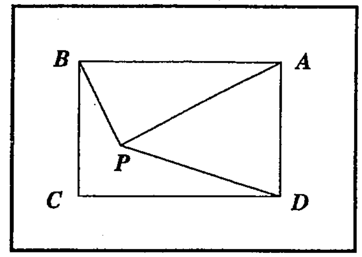
Contest #1 **October 15, 2007**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. How many real numbers are solutions to the equation, $x^4 + |x| = 10$?

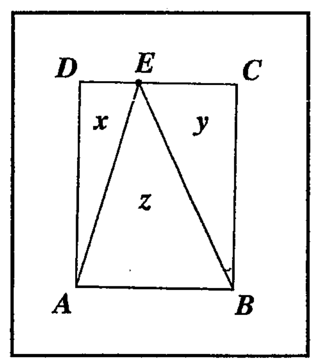
T-2. Let $ABCD$ be a rectangle and let P be a point inside the rectangle. If $PA = 8$, $PB = 4$, *and* $PD = 7$, then find the length of PC.



T-3. In the sum below, the letter F equals zero, and the other letters represent the digits, 1, 2, 3, 4, 5, or 6, with each digit used exactly once. If the 2-digit integer AB is a prime number, then what is the value of $A + B$?

$$\begin{array}{r} AB \\ +CD \\ \hline EFG \end{array}$$

T-4. Suppose that $ABCD$ is a rectangle, and that E is a point on $\overline{CD}$. Let x be the area of ΔAED, let y be the area of ΔBCE, and let z be the area of ΔABE. If $y^2 = xz$, what is the value of $\dfrac{DE}{EC}$?



T-5. Let x and y be real numbers.

Find all pairs (x, y) such that the sum $x + y$, the product xy, and the quotient $\dfrac{x}{y}$ are all equal.

MCWP Senior High Mathematics League

Contest 1 **October 15, 2007**

Individual Questions

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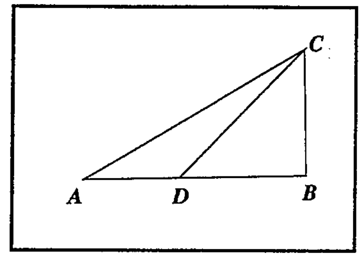
INDIVIDUAL ROUND 1 — 15 MINUTES

I–1. In the given figure, $\angle ABC = 90°$, $\angle CDB = 45°$, $\angle CAB = 30°$, *and* $AD = 2$. Find the length BC



I–2. How many points do the graphs of $4x^2 - 9y^2 = 36$ *and* $x^2 - 2x + y^2 = 15$ have in common?

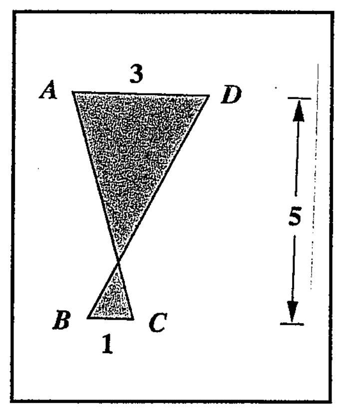
I–3. How many occurrences of the digit 5 are there in the list of numbers, 1, 2, 3, 4, · · · 1000 ?

--- CUT HERE ---

INDIVIDUAL ROUND 2 — 15 MINUTES

I–4. A drawer contains 64 socks. Each sock is one of 8 colors, and there are 8 socks of each color. If the socks in the drawer are thoroughly mixed and you randomly choose two of the socks, then what is the probability that these two socks will have the same color?

I–5. Suppose that $\overline{AD}$, $\overline{BC}$, $\overline{AC}$, *and* $\overline{BD}$ are line segments with line ***AD*** parallel to line ***BC***. If $AD = 3$, $BC = 1$, and the distance from line ***AD*** to line ***BC*** is equal to 5, then what is the sum of the areas of the two shaded triangles?



I–6. Let x and y be real numbers such that $\left(x^2 - y^2\right)\left(x^2 - 2xy + y^2\right) = 3$ and $x - y = 1$. What is the value of xy ?

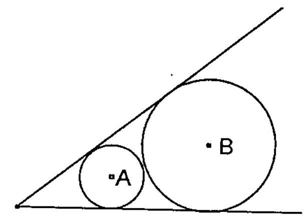
MCWP Senior High Mathematics League

Contest #2 **November 19, 2007**

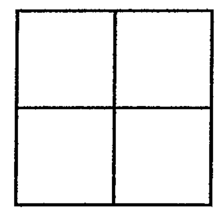
TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

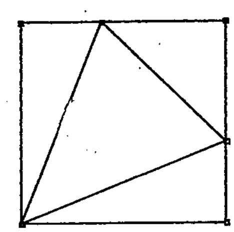
T-1 A circle with center A and radius 7 is tangent to the sides of an angle of $60°$. A larger circle with center B is tangent to the sides of the angle and to the first circle. Find the radius of the larger circle.



T-2. A square is partitioned into four congruent smaller squares by lines parallel to its sides. A nonzero digit is to be written in each smaller square, so that we get four two-digit numbers — two when you read across from left to right and two when you read down from top to bottom. The sum of these four two-digit numbers is 67. What are the four digits inscribed in the smaller squares?



T-3. An equilateral triangle is inscribed in a square. If each side of the square measures 1 inch, find the length of a side of the equilateral triangle.



T-4. How many positive integers less than 1000 have the sum of their digits equal to 6 ?

T-5. A grocer weighed four pieces of fruit: a lemon, a pear, an apple, and a grapefruit. The following observations were made:

(a) the lemon weighed more than the pear,
(b) the combined weight of the pear and the apple was greater than the combined weight of the lemon and the grapefruit, and
(c) the combined weight of the lemon and the pear was equal to the combined weight of the apple and the grapefruit.

List the four fruits in order of weight from heaviest to lightest.

MCWP Senior High Mathematics League

Contest #2 **INDIVIDUAL QUESTIONS** **November 19, 2007**

CUT HERE

INDIVIDUAL ROUND 1 — 15 MINUTES

1. What is the units' digit of the product 99π, when it's multiplied out?

2. The diameter of my car's new tires was 26 inches. When wore down uniformly 1/4 inch, the tires will need $x\%$ more revolutions to cover a given distance than when the tires were new. To the nearest tenth, what is the value of x?

3. Of the 24 possible arrangements of the digits 9, 4, 3, and 2, how many represent a prime?

CUT HERE

INDIVIDUAL ROUND 2 — 15 MINUTES

4. In a circle with both an inscribed and a circumscribed equilateral triangle, if the area of the smaller of these two triangles is 12, what is the area of the larger?

5. What are two numbers which differ by 5 whose square roots add up to 5?

6. If the coefficients of the cubic polynomial P are positive integers, for what value of k will the graph of $y = P(x)$ pass through $(1,10)$, $(3,k)$, and $(10,1234)$?

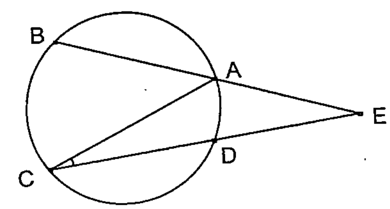
MCWP Senior High Mathematics League

Contest #3 **December 10, 2007**

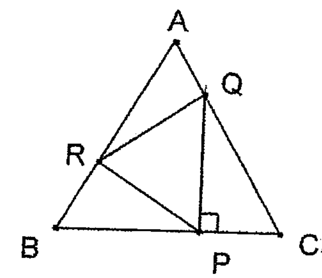
TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

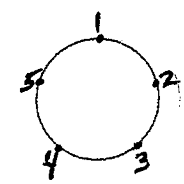
T-1. In a circle the arcs AB, BC, and CD are equal and each less than a half circle. Given that the measure of $\angle CEB = 40°$, find the measure of $\angle DCA$ in degrees.



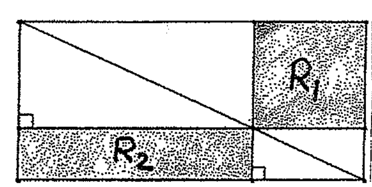
T-2. The equilateral triangle, ΔPQR, is inscribed in the equilateral triangle, ΔABC, as shown, with $\overline{PQ} \perp \overline{BC}$.
Find the ratio of the area of ΔPQR to the area of ΔABC.



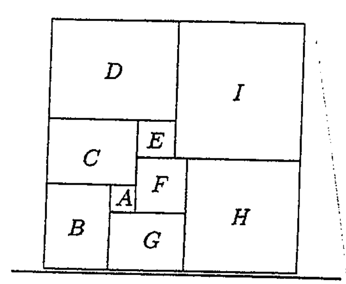
T-3. Five points on a circle are numbered 1, 2, 3, 4, and 5 in clockwise order as shown. A bug jumps in a **counterclockwise** direction from one point to another around the circle. If it is on an odd-numbered point, it moves two points, and if it is on an even-numbered point, it moves one point. If the bug begins on point 1, after 2007 jumps, what numbered point will it land on?



T-4. The area of the shaded rectangle R_1 is 1 sq. in.
Find the area of the shaded region R_2.



T-5. Nine labeled squares are arranged as shown. If square A has an area of $1 \ in^2$, and square B has an area of $81 \ in^2$, then what is the area in square inches of square I ?



MCWP Senior High Mathematics League

Contest #3 **INDIVIDUAL QUESTIONS** **December 10, 2007**

CUT HERE

INDIVIDUAL ROUND 1 — 15 MINUTES

1. A closed curve is drawn in a plane so that each point on the curve is 1 unit outside a 3×4 rectangle. What is the area of the region in the plane that's enclosed by this curve?

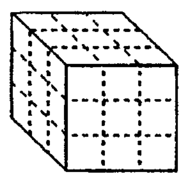
2. Of the first 1200 positive integers, how many are divisible by neither 3 nor 4?

3. How many teams are in a competition in which each match pits 2 teams against each other if it takes 105 matches for every team to play against every other team exactly once?

CUT HERE

INDIVIDUAL ROUND 2 — 15 MINUTES

4. A cubical solid is cut into 27 smaller cubical solids, as shown at the right. This can be done in as few as 6 slices by cutting through more than one piece at a time. For safety, pieces may be moved before or after (but not *during*) each slice. What is the least number of slices that are needed to cut a cubical solid into 64 smaller cubical solids?



5. In an equilateral triangle with both an inscribed and a circumscribed circle, if the area of the larger of these two circles is 60, what is the area of the smaller?

6. My truck's tires have a diameter of 2.5 feet. What is my truck's speed, in miles per hour, when its wheels turn 600 times per minute?

MCWP Senior High Mathematics League

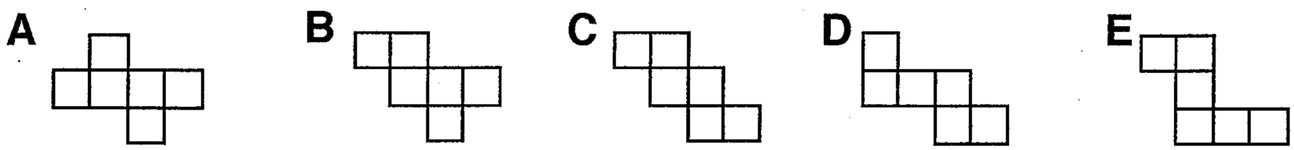
Contest #4 **January 14, 2008**

TEAM QUESTIONS

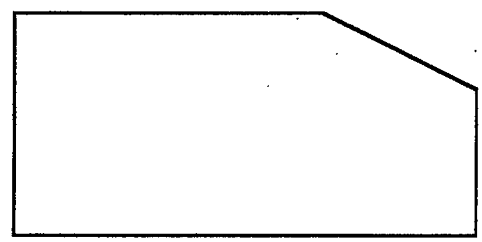
TIME LIMIT = 10 MINUTES

T-1. The sequence, $6, -9, x, y$ is such that the first three terms form an arithmetic sequence, and the last three terms form a geometric sequence. Find the values of x and y.

T-2. Which of the following configurations of six squares cannot be folded into a cube?



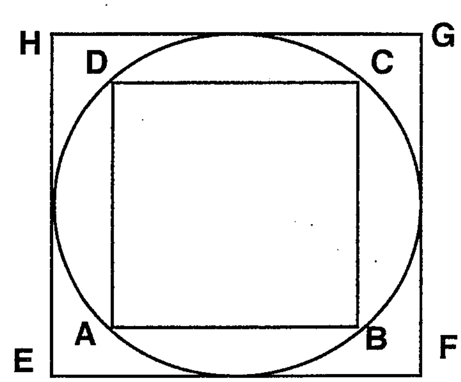
T-3. A pentagon is formed by cutting a triangular corner from a rectangular piece of paper, as shown. The five sides of the pentagon have lengths 13, 19, 20, 25 and 31, although this is not their order around the pentagon. What is the area of the pentagon?
NOTE: The diagram is not to scale.



T-4. A digit is placed in each empty square in the grid below so that each row contains each of the digits 1, 2, 3, 4, 5, and each column contains each of the digits, 1, 2, 3, 4, 5. What digit must be placed in the square at the bottom right corner of the grid?

	5	4		
1	3			
		5	3	
2		3	1	

T-5. The sum of the areas of inscribed square ABCD and circumscribed square EFGH is 108. What is the radius of the circle?



MCWP Senior High Mathematics League

Contest #4 **INDIVIDUAL QUESTIONS** **January 14, 2008**

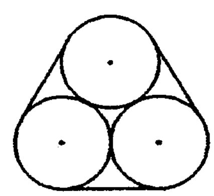
CUT HERE

INDIVIDUAL ROUND 1 — 15 MINUTES

1. I first drove 16 km at 48 km/hr, then I drove 20 km at 40 km/hr, and finally I drove 24 km at 36 km/hr. What was my average speed, in km/hr, for the entire trip?

2. How many different triangles have vertices selected from the seven points (–4,0), (–2,0), (0,0), (2,0), (4,0), (0,2), and (0,4)?

3. Three circular cylinders are strapped together as shown at the right. The cross-section of each cylinder is a circle of radius 1. Presuming that the strap used to bind the cylinders together has no thickness and no extra length, how long is the binding strap?



CUT HERE

INDIVIDUAL ROUND 2 — 15 MINUTES

4. Express, as a *fraction in lowest terms*, the sum

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\dots+\frac{1}{(n)(n+1)}+\dots+\frac{1}{99\times 100}.$$

5. Randomly select one of the 900 three-digit numbers in {100, 101, . . . , 998, 999}. What is the probability that the number selected is a perfect square?

6. What are the coordinates of the reflection of (6,0) across the graph of $y = 3x$?

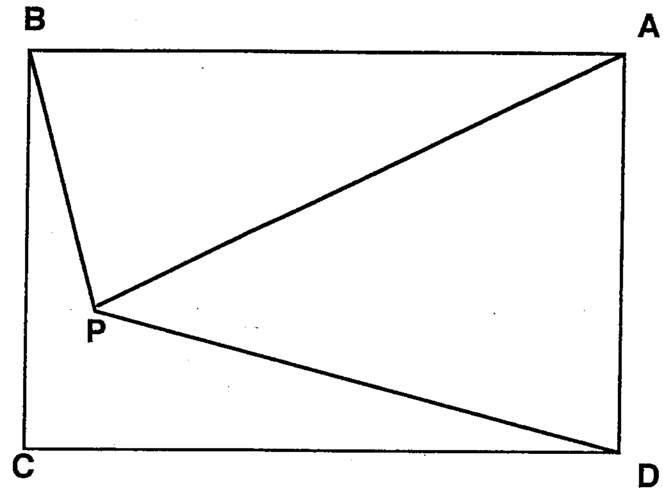
MCWP Senior High Mathematics League

Contest #5 **February 4, 2008**

TEAM QUESTIONS

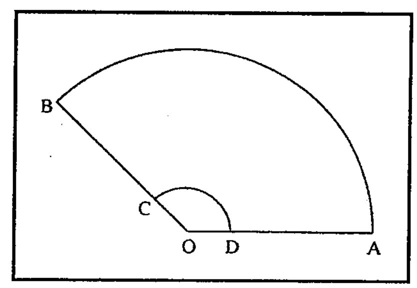
TIME LIMIT = 10 MINUTES

T-1. Let ABCD be a rectangle and let P be a point inside the rectangle. If $PA = 8$, $PB = 4$, *and* $PD = 7$, find the length PC.



T-2. Find all possible first terms of the geometric sequence if the sum and the product of its first three terms are equal to 63 and 1728, respectively.

T-3. In the figure shown, OAB and ODC are sectors of two concentric circles centered at O. The length of $\overline{OD}$ is one third the length of $\overline{OA}$. What is the ratio of the area of the region ABCD to the area of the sector OCD ?



T-4. A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jar is increased by 25% without altering the volume, by what percent must the height be decreased?

T-5. If $x + y = 1$ and $x^2 + y^2 = 4$, what is the value of $x^3 + y^3$?

MCWP Senior High Mathematics League

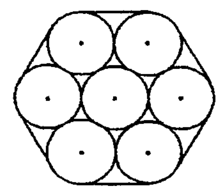
Contest #5 **INDIVIDUAL QUESTIONS** **February 4, 2008**

CUT HERE

INDIVIDUAL ROUND 1 — 15 MINUTES

1. Write the three three-digit perfect squares which, when taken together, use every non-zero digit exactly once.

2. Seven circular cylinders are strapped together as shown at the right. The cross-section of each cylinder is a circle of radius 1. Presuming that the strap used to bind the cylinders together has no thickness and no extra length, how long is the binding strap?

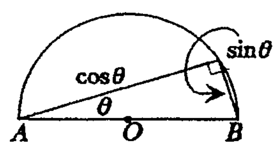


3. If $x \neq 0$ is real, then, for what value of k does $\dfrac{x^{n+2}+x^{n}}{x^{n+4}+x^{n+2}}$ simplify to x^k for all real n?

CUT HERE

INDIVIDUAL ROUND 2 — 15 MINUTES

4. In the diagram at the right, the semicircle's diameter has a length of 1. If the area of the semicircle is twice that of the right triangle, what is the measure of the triangle's smaller acute angle (to the nearest degree)?



5. What are all values of x for which $(x!)^2 - 7x! + 6 = 0$, where $x!$ represents x factorial?

6. Write a list, in increasing size order, of every four-digit number greater than 1000 which has four different digits. What is the 111th entry on this list?

MCWP Senior High Mathematics League

Contest #6 **March 3, 2008**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

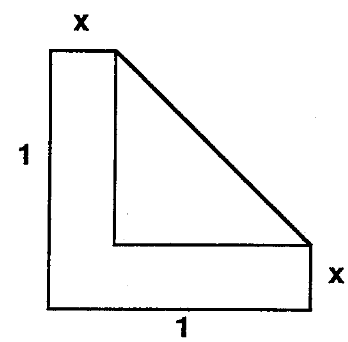
T-1. A car dealer sold two quality used cars for \$9999 each. On one she made a 10% profit, and on the other she had a 10% loss. What was her overall profit or loss for the two transactions combined ?

T-2. Some cookies had been eaten, without Mrs. Brown's permission, by one or more of her five children. When questioned, they gave the following answers:

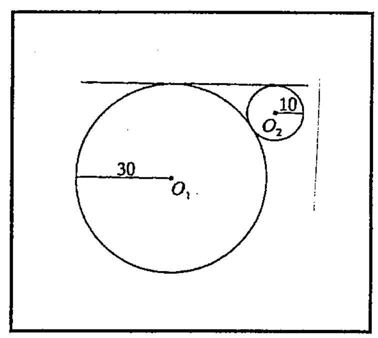
Alan: One of us ate the cookies.
Betty: Two of us ate the cookies.
Charles: Three of us ate the cookies.
Dee: Four of us ate the cookies.
Earl: All of us ate the cookies.

Mrs. Brown knew from the past behavior of her children that the guilty ones lied while the others told the truth. What was the number of children who ate the cookies?

T-3. In the diagram shown, the area of the letter L is equal to the area of the triangle. If the base and height of the L are both 1, then what is the length x of the ends of the L?



T-4. Two circles with centers O_1 *and* O_2 and radii of lengths 30 and 10, respectively, are tangent to each other and to a line as shown. Find the area of the shaded region bounded by the two circles and the line.



T-5. One gear turns $33\frac{1}{3}$ times per minute. Another gear turns 45 times per minute. Initially, a mark on each gear is pointing due north. After how many **<u>seconds</u>** will the two gears next have both their marks pointing due north?

MCWP Senior High Mathematics League

Contest #6 **INDIVIDUAL QUESTIONS** **March 3, 2008**

INDIVIDUAL ROUND 1 — 15 MINUTES

1. If the area of a circle's inscribed square is 60, what is the area of its circumscribed square?

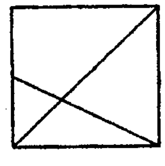
2. What are all values of x for which $\log_x \sqrt{x+12} > 1$?

3. Determine the units' digit of the sum $0! + 1! + 2! + \ldots + n! + \ldots + 20!$

INDIVIDUAL ROUND 2 — 15 MINUTES

4. What is the area of a trapezoid the lengths of whose bases are 10 and 16 and the lengths of whose legs are 8 and 10?

5. A diagonal of a square intersects a segment that connects one vertex of the square to the midpoint of an opposite side, as shown. If the length of the shorter section of the diagonal is 2, what is the area of the square?



6. What real value of x satisfies $\sqrt{5x} - \sqrt{2x} = 5 - 2$?

SOLUTIONS TO TEAM QUESTIONS

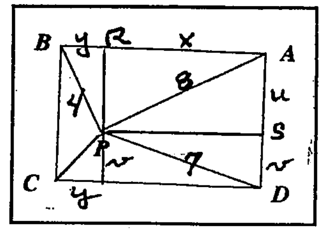
Contest #1, 2007 - 2008 **October 15, 2007**

T-1. The number of real roots of $x^4 + |x| = 10$ is the same as the number of intersection points of the graphs of $y = |x|$ and $y = 10 - x^4$. Is is easy to see that these two graphs intersect in exactly two points. Therefore, the are exactly two real roots for the given equation.

T-2. Draw $\overline{PR}$ perpendicular to $\overline{AB}$, and draw $\overline{PS}$ perpendicular to $\overline{AD}$. Let $AR = x,\ BR = y,\ AS = u,\ and\ DS = v$. The conditions of the problem imply that $y^2 + u^2 = PB^2 = 4^2 = 16$, $u^2 + x^2 = PA^2 = 8^2 = 64$, and $x^2 + v^2 = PD^2 = 7^2 = 49$. Therefore,

$$PC^2 = y^2 + v^2 = 16 - u^2 + 49 - x^2 = 65 - \left(x^2 + u^2\right) = 65 - 64 = 1.$$

Therefore, $PC = 1$.



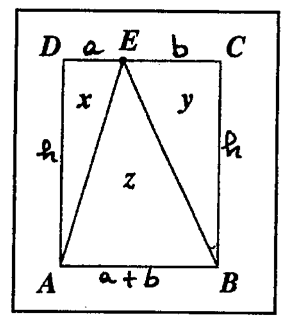
T-3. Since AB is prime, B is either 1 or 3. Also, $A + C \le 11$, so $E = 1$. It follows that $B = 3$. Since $D \le 6$, then $B\_D < 10$, and $A + C = 10$. We can then determine that A and C must be 4 and 6 in some order. Since 63 is not prime, we have AB = 43, and the sum of the digits is 7.

T-4. Let $a = DE,\ b = EC,\ and\ h = AD = BC$.

Then, area $x = \frac{1}{2}ah$, area $y = \frac{1}{2}bh$, and area $z = \frac{1}{2}(a+b)h$

Since $y^2 = xz$, we have $\left(\frac{1}{2}bh\right)^2 = \left(\frac{1}{2}ah\right)\left(\frac{1}{2}(a+b)h\right)$

from which $\frac{1}{4}b^2h^2 = \frac{1}{4}a(a+b)h^2 \Rightarrow b^2 = a^2 + ab$.



Now divide through by b^2 to get $1 = \left(\frac{a}{b}\right)^2 + \frac{a}{b} \Rightarrow \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 1 = 0$.

Solving this quadratic equation for $\left(\frac{a}{b}\right)$, we find: $\left(\frac{a}{b}\right) = \frac{-1 \pm \sqrt{1+4}}{2}$, so

Since the required root must be positive, we have $\frac{DE}{EC} = \frac{a}{b} = \frac{-1+\sqrt{5}}{2}$

T-5. From $xy = \frac{x}{y}$ we see that $y \neq 0$, and if $x \neq 0$, then $y^2 = 1 \Rightarrow y = -1,\ 1$.

If $y = 1$, then $x + y = xy$ becomes $x + 1 = x$ which has no solution.

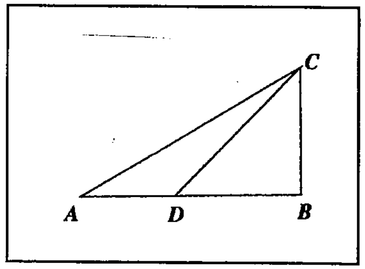
If $y = -1$, then $x + y = xy$ becomes $x - 1 = -x$, and $x = \frac{1}{2}$.

Then the required ordered pair is $\left(\frac{1}{2}, -1\right)$

SOLUTIONS TO INDIVIDUAL QUESTIONS

Contest #1, 2007-2008

I-1. Let $x = BC$. Since ΔDBC is a 45-45-90 right triangle, $DB = x$.



Since ΔABC is a 30-60-90 right triangle, and $\tan(30°) = \frac{1}{\sqrt{3}}$, we can determine that

$$\frac{1}{\sqrt{3}} = \tan(30°) = \frac{BC}{AB} = \frac{BC}{AD + DB} = \frac{x}{2 + x}.$$

Therefore, $2 + x = x\sqrt{3}$ *and* $BC = x = \frac{2}{\sqrt{3} - 1} = \sqrt{3} + 1$

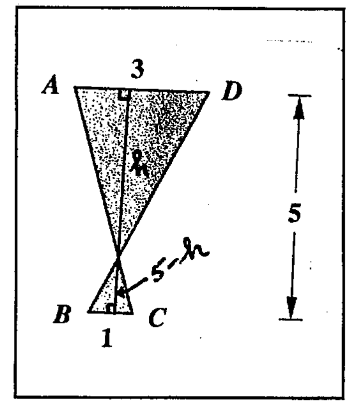
I-2. The graph of $4x^2 - 9y^2 = 36$ is an hyperbola with vertices at $(-3,0)$ *and* $(3,0)$. The graph of $x^2 - 2x + y^2 = 15$ is a circle of radius 4 centered at $(1,0)$, and therefore crossing the x-axis at $(-3,0)$ *and* $(5,0)$. It follows that these graphs intersect at 3 points.

I-3. Consider each of the 1000 non-negative integers less than 1000 as a 3-digit number: $0 = 000,\ 1 = 001,\ 2 = 002,\ \cdots,\ 999$.
Then each digit (0 through 9) occurs an equal number of times.
Since the total number of resulting digits in the 3-digit numbers is $3 \times 1000 = 3000$, each digit occurs 300 times.

October 15, 2007

I-4. After the first sock is chosen, there are 63 socks left, 7 of which are the same color as the first sock. Therefore, the probability is $\frac{7}{63} = \frac{1}{9}$

I-5. The two shaded triangles are similar. Therefore the ratio of their altitudes is equal to the ratio of the lengths of the bases of the triangles. Let h be the altitude of the larger triangle. Then $5 - h$ is the altitude of the smaller triangle.



Now, the sum of the areas is $\frac{1}{2}(3)(h) + \frac{1}{2}(1)(5 - h) = h + \frac{5}{2}$.

But, from the similar triangles, we have $\frac{3}{1} = \frac{h}{5 - h}$, so $h = \frac{15}{4}$.

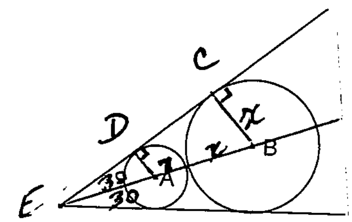
Then the sum of the areas is $h + \frac{5}{2} = \frac{15}{4} + \frac{5}{2} = \frac{25}{4} = 6.25$

I-6. From the given information,
$3 = \left(x^2 - y^2\right)\left(x^2 - 2xy + y^2\right) = (x - y)(x + y)(x - y)^2 = x + y$

Therefore, $2x = (x + y) + (x - y) = 3 + 1 = 4$, and
$2y = (x + y) - (x - y) = 3 - 1 = 2$.
It follows that $x = 2$ and $y = 1$, so then $xy = 2$.

T-1 Note that ΔADE and ΔBCE are similar 30°–60°–90° triangles. Then since $AD = 7$, we have $EA = 14$. Then if x is the radius of the larger circle, we have the proportion,

$$\frac{x}{21 + x} = \frac{7}{14} = \frac{1}{2}, \text{ and } x = 21$$



T-2. For a sum of 67, we can't have more than 1 number beginning with 2. Consider having 2 numbers beginning with a 2. Then we'd have something like this, for which the sum of the four numbers is greater than 67. Now consider having only one 2 in the grid.

2	2
1	x

1	2
1	x

Then we'd have $12 + (10 + x) + 11 + (20 + x) = 67 \Rightarrow 53 + 2x = 67$, so that $x = 7$.

Therefore the numbers could be arranged

1	2
1	7

or

1	1
2	7

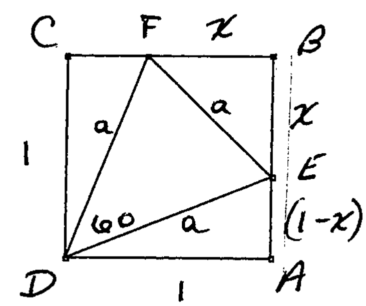
T-3. By the hypotenuse–leg theorem, we see that $\Delta DCF \cong \Delta DAE$. Therefore, $CF = AE$. Let a be the length of one side of the equilateral triangle and let $FB = BE = x$. Then $CF = AE = (1 - x)$. By the Pythagorean Theorem, in ΔDAE, $a^2 = 1 + (1 - x)^2$. In ΔFBE, $a^2 = x^2 + x^2 = 2x^2$, so $x = \frac{a}{\sqrt{2}}$.

Substituting this into the first equation, we have

$$a^2 = 1 + (1 - x)^2 = 1 + \left(1 - \frac{a}{\sqrt{2}}\right)^2 = 1 + 1 - \sqrt{2}a + \frac{a^2}{2} \text{ which yields}$$

$$a^2 + 2\sqrt{2}a - 4 = 0. \text{ Solving, we have } a = \frac{-2\sqrt{2} \pm \sqrt{8 + 16}}{2} = -\sqrt{2} \pm \sqrt{6}$$

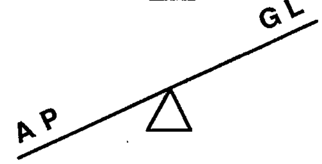
Since $a > 0$, $a = \sqrt{6} - \sqrt{2}$ *OR EQUIVALENTLY*, $2\sqrt{2 - \sqrt{3}}$



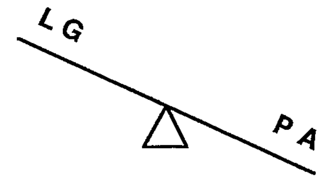
T-4. Look for a pattern. Of the numbers from 1 to 99, seven numbers have this property — 6, 15, 24, 33, 42, 51, 60. Of the numbers from 100 to 199, six numbers have this property — 105, 114, 123, 132, 141, 150. Of the numbers from 200 to 299, five have this property — 204, 213, 222, 231, 240. Following this pattern, there will be 4 numbers in the 300's, 3 numbers in the 400's, 2 numbers in the 500's and 1 number in the 600's that fit this criteria. Therefore the required solution is $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$.

T-5. Let the weights of the fruits be represented by the letters, A, G, L, and P. We know that **L > P**. Consider a balance scale which will balance with G and A on one side and L and P on the other side, like this:

We are told that (P + A) > (L + G). If we think of this as switching the positions of the lemon and apple, the balance will look like this. So, we can conclude that A > L.



Alternatively, we can think of this as switching the locations of the pear and the grapefruit, and the balance will then look like this. So, we can conclude that P > G.



Put these two results together with the given information that L > P, and we have: **A > L > P > G**

Math Council of Western PA Sr High Math Leauge 2007-2008
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #2 — November 19, 2007

Answers

1. 1   -2. 2.0 (Do *NOT* accept 2)   3. 0

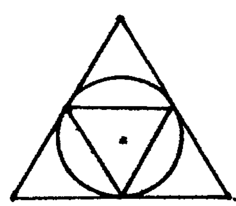
4. 48   5. 4,9   6. $58 = 1(3)^3+2(3)^2+3(3)+4$

1) Since $\frac{22}{7}$ is a better approximation to π than 3.14, it follows that $\frac{22}{7} \times 99 = \frac{2178}{7} = 311.14\ldots$ is closer to the exact value than is $99 \times 3.14 = 310.86$. Therefore $99\pi > 311$, slightly, so the units' digit is $\boxed{1}$.

2) The ratio of tire circumferences is the same as the ratio of number of revolutions needed to cover a fixed distance. Since $\frac{26}{25.5} = 1.0196\ldots \approx \boxed{2.0}\%$ more when we round off 1.96%.

3) The sum of the digits is 9, so every arrangement is divisible by 9. Therefore, the number of primes is $\boxed{0}$.

4) Since the diagram is self-explanatory the ratio is 4:1, so $4(12) = \boxed{48}$.



5) The result is easy to see by inspection. Since the sum of the square roots is 5, it's easy to guess. More formally, one can solve $\sqrt{x} + \sqrt{x+5} = 5$ to get $x = 4$ and $x+5 = 9$: $\boxed{4,9}$.

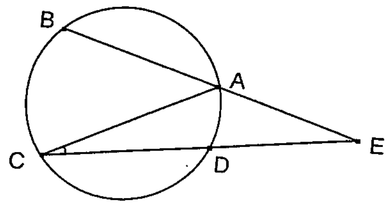
6) $P(x) = ax^3 + bx^2 + cx + d$, where a,b,c,d are positive integers. $P(1) = a+b+c+d = 10$; $P(10) = 1000a + 100b + 10c + d = 1234$. From the equation $a+b+c+d = 10$ we know $1 \le a,b,c,d \le 7$. Therefore $a = 1, b = 2, c = 3, d = 4$ are only integers in this interval for which $P(10) = 1234$. Finally $P(3) = 1(3)^3 + 2(3)^2 + 3(3) + 4 = \boxed{58}$.

SOLUTIONS TO TEAM QUESTIONS

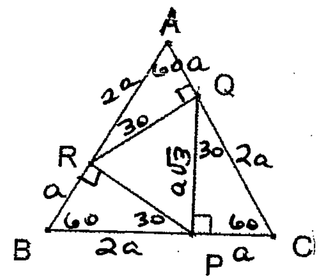
Contest #3, **December 10, 2007**

T-1. Refer to the diagram. We know that $\frac{1}{2}[\overset{\frown}{BC} - \overset{\frown}{AD}] = \angle BEC = 40°$, so $\overset{\frown}{AD} = \overset{\frown}{BC} - 80°$. But, since the arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CD}$ are all equal, we can see that $\overset{\frown}{AD} = 360° - 3(BC)$. Therefore, $\overset{\frown}{AD} = \overset{\frown}{BC} - 80° = 360° - 3(\overset{\frown}{BC})$ and arc $\overset{\frown}{BC} = 110°$. But, we also know that $\angle ACD = \frac{1}{2}(\overset{\frown}{AD})$.
Substituting, we have

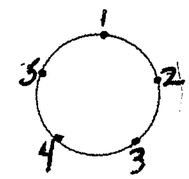
$$\angle DCA = \frac{1}{2}(\overset{\frown}{AD}) = \frac{1}{2}(\overset{\frown}{BC} - 80°) = \frac{1}{2}(110° - 80°) = 15°$$



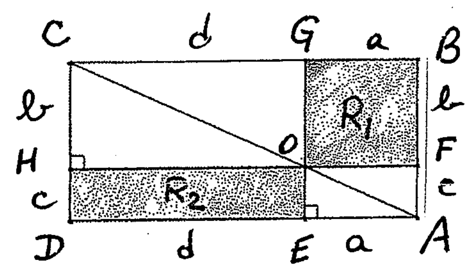
T-2. The area of an equilateral triangle with side s is given by $\frac{s^2\sqrt{3}}{4}$.
By computing the angles, we see that we have three 30°–60°–90° triangles — ΔARQ, ΔCQP, *and* ΔPBR. By the Hypotenuse - Angle Theorem, these 3 triangles are congruent. Let $a = PC = QA = RB$, then $CQ = AR = BP = 2a$.
Then the length of one side of the inscribed triangle is $a\sqrt{3}$, and the length of one side of the larger triangle is $3a$. Therefore, the areas are in the ratio of the squares of the sides of the triangles, or $\frac{(a\sqrt{3})^2}{(3a)^2} = \frac{3a^2}{9a^2} = \frac{1}{3}$



T3. The problem states that the bug moves <u>counter</u>clockwise. Therefore, if it starts at 1, on the first jump it moves to 4, then to 3, then to 1, and then repeats this pattern. Therefore if the number of jumps, n, is a multiple of 3, the bug will be on the **1** after the n-th jump.
Since $2007 = 3(669)$, after 2007 jumps, the bug will be on the **1**.

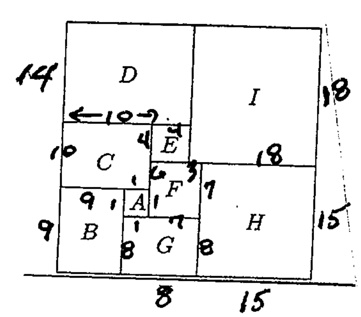


T-4. The area of $R_1 = ab$, and the area of $R_2 = cd$.
We notice similar triangles, $\Delta CHO \approx \Delta OEA$.
Therefore, the corresponding sides are proportional and $\frac{b}{d} = \frac{c}{a}$.
Then $ab = cd = 1$, and the area of region $R_2 = 1$



T-5. This is just a matter of carefully determining the lengths of the sides of the squares beginning with the given information. Square A has side 1, square B has side 9, so square G h as side 8, then square F has side 7, then square C has side 10, then square E has side 4, then square D has side 14, and then square I has side 18.

The length of square ***I*** is 18, so the area of square ***I*** is 324 square inches.



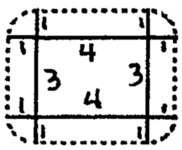
Math Council of Western PA Sr High Math Leauge 2007 - 2008
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #3 — December 10, 2007

Answers

1. $26+\pi \approx 29.14159265359\ldots$ 2. 600 3. 15

4. 6 5. 15 6. $375\pi/22 \approx 53.549874777\ldots$ or $(375\pi/22)$ mph

1)



The boundary is the dotted line. The quarter-circles have radius-length 1. The area is the sum of the areas of 4 quarter-circles + 5 rectangles = $4\left(\frac{\pi r^2}{4}\right) + 2\times(1\times4) + 2\times(1\times3) = \pi + 8 + 6 + 12 = \boxed{26+\pi}$.
$+(4\times3)$

2) (# divisible by either or both) = (# div by 3) + (# divisible by 4) − (# div by 12)
$= \left[\frac{1200}{3}\right] + \left[\frac{1200}{4}\right] - \left[\frac{1200}{12}\right] = 400 + 300 - 100$
$= 600$

divisible by neither $= 1200 - 600 = \boxed{600}$.

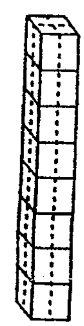
3) $\binom{n}{2} = 105 \Leftrightarrow \frac{(n)(n-1)}{(2)(1)} = 105 \Leftrightarrow n^2 - n - 210 = 0$
$\Leftrightarrow (n+14)(n-15) = 0$
$n = \boxed{15}$.

4) If each slice doubles the number of slices, we can create $2^6 = 64$ sliced pieces in $\boxed{6}$ slices. Can we make them all cubes?

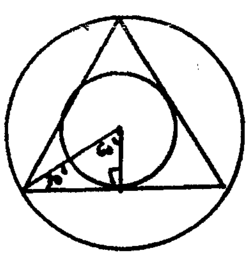
slice 1
slice 2
slice 3

We now have 8 cubes. Make the 2 slices shown in the diagram at far right, on this line.

We now have 32 $1\times1\times2$ solids – that look like 32 bricks. Line them up so 1 more slice creates 64 cubes.



5)



Ratio of radii = 2 to 1.
Ratio of areas = $\left(\frac{2}{1}\right)^2 = 4$ to 1
Area of smaller $= \frac{60}{4} = \boxed{15}$.

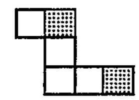
6) Each minute, distance covered by a tire $= (600)\left(\frac{5\pi}{2}\right) = 1500\pi$
In an hour, a tire covers a distance of $60 \times 1500\pi$ feet. The car's speed, in mph, is $\frac{60 \times 1500\pi \text{ feet}}{5280 \text{ feet/mile}} = \boxed{\frac{375\pi}{22}}$ mph

SOLUTIONS TO TEAM QUESTIONS

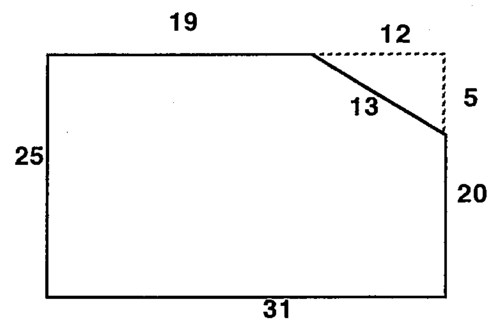
Contest #4, 2007 - 2008 **January 14, 2008**

T-1. Since $6, -9, x$ is an arithmetic sequence, the common difference is -15 so $x = -9 + (-15) = -24$.
Then since $-9, -24, y$ is a geometric sequence, the common ratio is $\frac{-24}{-9} = \frac{8}{3}$, so $y = (-24)\left(\frac{8}{3}\right) = -64$.
So, $x = -24$ and $y = -64$.

T-2. This is simply a problem of visualization. Choice E is the only one that cannot be folded into a cube, since when folded, the two shaded squares would overlap.



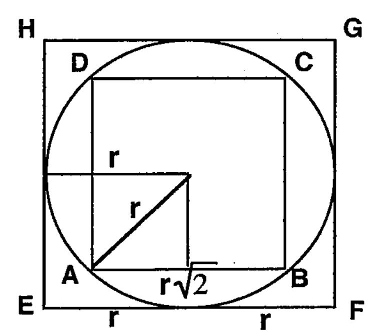
T-3. The triangular piece that was removed is a right triangle. Consider possible right triangles with integer sides. We can see that if this is a 5–12–13 right triangle, the given numbers fit around the pentagon as shown. The area of the triangle removed is $(.5)(12)(5) = 30$. The area of the original rectangle is $(25)(31) = 775$. So, the area of the pentagon is $775 - 30 = 745$ square units.



T-4. For discussion, let (4,2) designate the location row 4, column 2, etc.
(More than 1 routine may be followed to achieve the required answer.)
We see the row 3 needs the numbers 1, 2, and 4.
The 4 must be at (3,1).
Column 1 then needs the numbers 3 and 5.
The 3 must be at (1,1) so the 5 is at (5,1).
Column 3 then needs the numbers 1 and 2.
The 2 must be at (2,3) so the 1 is at (5,3).
Row 1 now needs the numbers 1 and 2.
The 2 must be at (1,4) so the 1 is at (1,5).
Column 4 now needs the numbers 4 and 5. The 4 must be at (5,4) so the 5 is at (2,4).
Row 5 now needs the numbers 2 and 3. The 2 must be at (5,2) so the 3 is at (5,5),
and the number in the bottom right corner is a 3.

	5	4		
1	3			
		5	3	
2		3	1	

T-5. Because we are working with squares, we see that
$\left(\frac{1}{2}\right)AB = \frac{r}{\sqrt{2}} \Rightarrow AB = \frac{2r}{\sqrt{2}} = r\sqrt{2}$.
We can also see that $EF = 2r$.
Since the sum of the areas of the squares is 108, we have
$\left(r\sqrt{2}\right)^2 + (2r)^2 = 108 \Rightarrow 2r^2 + 4r^2 = 108 \Rightarrow r^2 = 18$
and $r = 3\sqrt{2}$



Math Council of Western PA Sr High Math Leauge 2007 - 2008
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #4 — January 14, 2008

Answers

1. 40 or 40 km/hr	2. 24	3. $6+2\pi$
4. 99/100	5. 11/450	6. (−4.8,3.6)

1)

D	R	T
16 km	48 km/hr	20 mins.
20 km	40 km/hr	30 mins.
24 km	36 km/hr	40 mins.

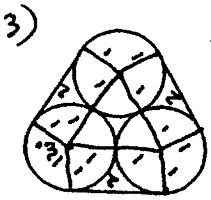
Total Time = 1.5 hrs
Total Distance = 60 km
$\therefore$ $\boxed{40}$ km/hr.

2) 

7 points

$\binom{7}{3} - \binom{5}{3} - \binom{3}{3} = 35-10-1 = \boxed{24}.$

$\binom{5}{3}$: 3-tuples all on x-axis; $\binom{3}{3}$: triples all on y-axis

3) 

Length = circum of 3 120° arcs + 3×2
= circum of 1 circle + 6 = $\boxed{2\pi+6}$.

4) $\frac{1}{(n)(n+1)} = \frac{n+1}{(n)(n+1)} - \frac{n}{(n)(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, so

$\frac{1}{1\times 2} = 1-\frac{1}{2}$

$\frac{1}{2\times 3} = \frac{1}{2}-\frac{1}{3}$

etc

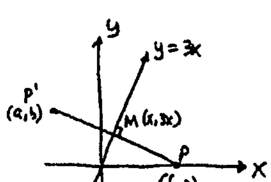
Sum $= \frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{98}-\frac{1}{99}+\frac{1}{99}-\frac{1}{100} = 1-\frac{1}{100} = \boxed{\frac{99}{100}}$.

5) $31^2=961$ and $32^2=1024$, so 1^2-31^2 are the 31 perfect squares ≤ 999

$9^2=81$, $10^2=100$, so 1^2-9^2 are the 9 perfect squares ≤ 99

$31-9 = 22$ perfect squares from 100 − 999

So prob $= \frac{22}{900} = \boxed{\frac{11}{450}}$.

6) 

Method I:

Since $m_{\overline{MP}} = -\frac{1}{3}$, $\frac{3x-0}{x-6} = -\frac{1}{3}$, so $x=\frac{3}{5}$

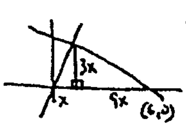
$3x = \frac{9}{5}$

M is the midpt of $\overline{P'P}$. Therefore,

$\left.\begin{array}{l}\frac{a+6}{2} = \frac{3}{5} \Rightarrow a = \frac{-24}{5} \\ \frac{b+0}{2} = \frac{9}{5} \Rightarrow b = \frac{18}{5}\end{array}\right\} \Rightarrow (a,b) = \boxed{\left(\frac{-24}{5}, \frac{18}{5}\right)}$

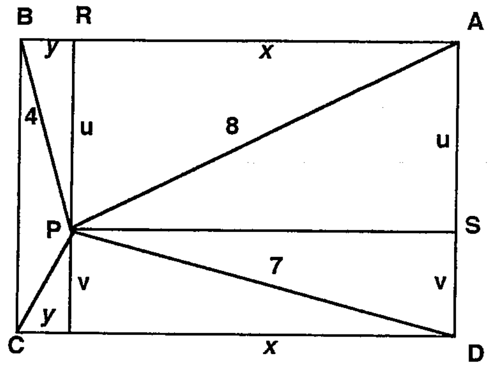
Method II:

By ~ △,
$10x=6$, so $x=3/5$
Continue as in Method I.



SOLUTIONS TO TEAM QUESTIONS

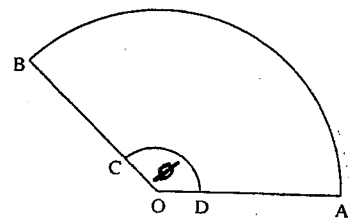
Contest #5, 2007 - 2008 **February 4, 2008**



T-1. Drop a perpendicular from P to side AB and call the intersection point R.
Set AR = x and BR = y. Similarly, drop a perpendicular from P to AD and call the intersection point S. Set AS = u, and DS = v.
Then $y^2 + u^2 = PB^2 = 16$, $u^2 + x^2 = PA^2 = 64$,
and $x^2 + v^2 = PD^2 = 49$. The required length is PC, and $PC^2 = y^2 + v^2$. Now, if we add the 3 equations above, we have, $y^2 + v^2 + 2x^2 + 2u^2 = 129$. Since $u^2 + x^2 = 64$, we have $y^2 + v^2 + 128 = 129$, so that $y^2 + v^2 = 1$. Therefore, $PC = 1$.

T-2. Let a be the first term and r be the ratio of the sequence. Then $a + ar + ar^2 = 63$ and $(ar)^3 = 1728$. Then $ar = 12$, and so $a + 12 + \dfrac{12^2}{a} = 63$, which leads to the equation, $a^2 - 51a + 144 = 0$. Solving we have, $(a-48)(a-3) = 0$, and the solutions are $a = 48$ *and* $a = 3$. So $r = \dfrac{12}{48} = \dfrac{1}{4}$ *or* $r = \dfrac{12}{3} = 4$.
So, the possible first terms are $a = 3$ *or* $a = 48$.

T-3. Refer to the labeled diagram. The area of sector OAB is $\left(\dfrac{\phi}{360}\right)(\pi)\left(OA^2\right)$
and the area of sector ODC is $\left(\dfrac{\phi}{360}\right)(\pi)\left(OD^2\right)$.



The required ratio is then $\dfrac{\left(\dfrac{\phi}{360}\right)(\pi)\left(OA^2 - OD^2\right)}{\left(\dfrac{\phi}{360}\right)(\pi)\left(OD^2\right)}$.

Since $OA = 3(OD)$ this ratio becomes $\dfrac{(3(OD))^2 - \left(OD^2\right)}{OD^2} = \dfrac{9-1}{1} = 8$

T-4. The volume of the original jar is $V = \pi r^2 h_1$, where r is the radius and h_1 is the original height.
Increase the radius by 25% to obtain $V = \pi\left(r + \dfrac{1}{4}r\right)^2 h_2 = \pi\left(\dfrac{5}{4}r\right)^2 h_2 = \left(\dfrac{25\pi}{16}\right)r^2 h_2$.
Then we need to have $\pi r^2 h_1 = \left(\dfrac{25\pi}{16}\right)r^2 h_2$, so that $h_1 = \left(\dfrac{25}{16}\right)h_2$ and $h_2 = \left(\dfrac{16}{25}\right)h_1 = (64\%)h_1$.
Therefore, the height would need to be reduced by 36%.

T-5. Squaring the first equation, we have $(x+y)^2 = 1^2 \Rightarrow x^2 + 2xy + y^2 = 1$.
Subtracting the second equation from this result we have, $2xy = -3$. Then, $xy = \dfrac{-3}{2}$.
Cubing both sides of the first equation, we have $1^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x+y)$.
Solving for $x^3 + y^3$, we have $x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 1 - 3\left(\dfrac{-3}{2}\right)(1) = 1 + \dfrac{9}{2} = \dfrac{11}{2}$.

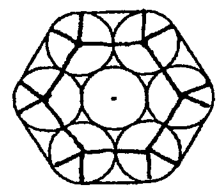
Math Council of Western PA Sr High Math Leauge 2007 - 2008
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #5 — February 4, 2008

Answers

1. 361, 529, 784	2. $12+2\pi$	3 -2
4. 26 or 26°	5. 0, 1, 3	6. 1297

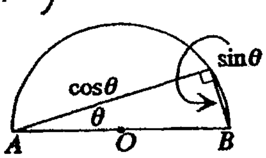
1) Here are all 3-digit squares whose digits are all different:
169, 196, 256, 289, 324, 361, 529, 576, 625, 729, 784, 841, 961
$\boxed{361, 529, 784}$

2) length = perim of hexagon + circumf of $\odot$*
$= 6(2) + 2\pi = \boxed{12+2\pi}$.
*each arc is a 60° arc, so six of them, when their lengths are added, will equal the circumference.



3 $\dfrac{x^{n+2}+x^n}{x^{n+4}+x^{n+2}} = \dfrac{x^n(x^2+1)}{x^{n+2}(x^2+1)} = \dfrac{x^n}{x^{n+2}} = x^{\boxed{-2}}$.

4)



Area of Δ $= \frac{1}{2}bh = \frac{1}{2}(\sin\theta)(\cos\theta)$; Area of semicircle $= \frac{\pi r^2}{2} = \frac{\pi}{8}$
Setting equation: Area semicircle = 2(Area of rt Δ)
$\frac{\pi}{8} = \sin\theta\cos\theta$
Thus, $2\sin\theta\cos\theta = \sin 2\theta = \frac{\pi}{4}$, so $2\theta = \text{Arcsin}\frac{\pi}{4}$, $\theta = \frac{1}{2}\text{Arcsin}\frac{\pi}{4}$
$\approx \boxed{26°}$

5) $(x!)^2 - 7x! + 6 = (x!-6)(x!-1) = 0$
$x! = 6 \quad x! = 1$
$x = 3 \quad x = 0, 1$ $\quad\boxed{0, 1, 3}$

6) The least such number is 1023. How many numbers begin $10\underline{?}\,\underline{?}$? Well, $10\underline{a}\,\underline{b}$ has 8 ways to choose "a" and 7 to choose "b" ⇒ 56 such numbers begin with $10\underline{?}\,\underline{?}$. Now count $\underbrace{1203 \to 1298}_{56\text{ nums}}$, so 112 so far. Thus, the 111^{th} entry is $\boxed{1297}$.

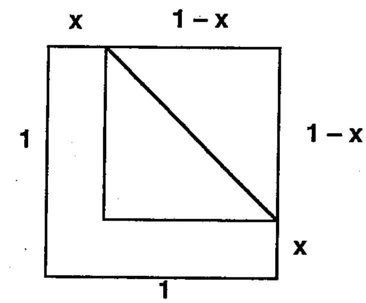
SOLUTIONS TO TEAM QUESTIONS

Contest #6, 2007 - 2008 **March 3, 2008**

T-1. Let the dealer cost of the vehicles be C_1 *and* C_2. Then the one sold at a 10% profit was sold at a price of $(1.1)C_1 = \$9,999$ and the one sold at a 10% loss was sold at a price of $(0.9)C_2 = \$9,999$. So, $C_1 = \$9,090$ and $C_2 = \$11,110$. Total revenue minus total cost yields $(2)(9999) - (9050 + 11110) = -202$. Therefore, there is an overall \$202 loss on the two transactions.

T-2. The number of children who ate the cookies must equal the number of liars. Since only one of the 5 statements can be true, there must be 4 liars. Therefore, 4 of the children ate the cookies.

T-3. Let x be the thickness of the L. Then the legs of the triangle each have length $(1-x)$. The area of the square containing the L is 1. We can see that the 2 triangles in the diagram have equal areas. Since a triangle and the L-shape must have equal areas, the area of the triangle must be 1/3.

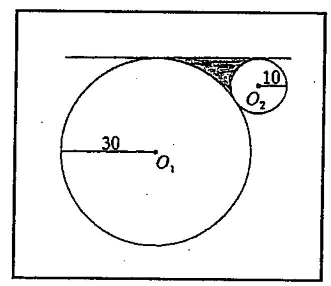


Then, for the area of a triangle, we have, $\frac{1}{2}(1-x)(1-x) = \frac{1}{3}$,

so that $(1-x)^2 = \frac{2}{3} \Rightarrow 1-x = \frac{\sqrt{6}}{3} \Rightarrow x = 1 - \frac{\sqrt{6}}{3}$

T-4. Draw radii from the center of each circle perpendicular to the line.
We see that AC = 20, CD = 10, and AB = 40. Therefore, ΔABC is a 30°-60°-90° triangle, so that $DE = 20\sqrt{3}$, and $m\angle DAB = 60°$ and $m\angle ABE = 120°$. The required area is the area of quadrilateral ABED minus the area of the 2 sectors of circles A and B: $\left(20\sqrt{3}\right)\left(\frac{30+10}{2}\right) - \left(\frac{60}{360}\right)(\pi)\left(30^2\right) - \left(\frac{120}{360}\right)(\pi)\left(10^2\right)$

$= 400\sqrt{3} - \left(\frac{1}{6}\pi\right)\left(30^2 + 2\left(10^2\right)\right) = 400\sqrt{3} - \frac{1100\pi}{6} = 400\sqrt{3} - \frac{550\pi}{3}$



T-5. The correct answer is 36 seconds.

Gear 1 has 1 revolution in $\frac{3}{100}$ minutes, or 1 revolution in $\left(\frac{3}{100}\right)60 = \frac{9}{5} = \frac{27}{15}$ seconds.

Gear 2 has 1 revolution in $\frac{1}{45}$ minutes, or 1 revolution in $\left(\frac{1}{45}\right)60 = \frac{4}{3} = \frac{20}{15}$ seconds.

Gear 1 has 20 revs in $\frac{(27)(20)}{15} = 36$ seconds, and Gear 2 has 27 revs in $\frac{(20)(27)}{15} = 36$ seconds.

To see that 36 is minimal, we want N revolutions for Gear 1 and M revolutions for Gear 2 such that $\frac{27}{15}N = \frac{20}{15}M$. Since 27 and 20 are relatively prime, N = 20, and M = 27.

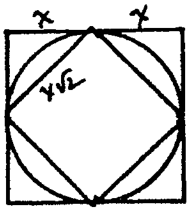
Math Council of Western PA Sr High Math Leauge 2007 - 2008
SOLUTIONS TO INDIVIDUAL QUESTIONS
Contest #6 — March 3, 2008

Answers

1\. 120   2. (1,4) or $\{x|1 < x < 4\}$ or equivalent conjunction   3. 4

4\. 104   5. 18   6. $7 + 2\sqrt{10}$

1)



$\frac{\text{area of circumsquare}}{\text{area of insquare}} = \left(\frac{2x}{x\sqrt{2}}\right)^2 = \frac{4}{2} = \frac{2}{1}$, so

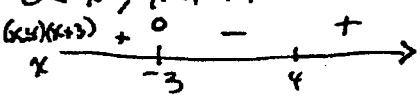
the area of the circumscribed square $= 2(60) = \boxed{120}$.

2) $\log_x \sqrt{x+12} > 1$ First, since the base is x, $0 < x$, $x \neq 1$.

If $x > 1$, then $\sqrt{x+12} > x$
$x + 12 > x^2$
$x^2 - x - 12 < 0$
$(x-4)(x+3) < 0$

If $0 < x < 1$, $\sqrt{x+12} < x$
$\vdots$
$x^2 - x - 12 > 0$
no solutions

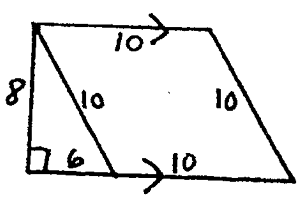
$(x-4)(x+3)$



$\boxed{\{x|1<x<4\}}$ or $(1,4)$

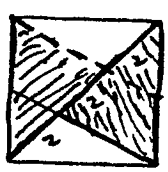
3) $0! = 1$ $1! = 1$ $2! = 2$ $3! = 6$ $4! = 24$ $5! = 120$. Whatever we multiply by now, the units' digit remains 0. So the sum has the same units' digit as $0!+1!+2!+3!+4! = 34$ whose units' digit is $\boxed{4}$.

4)



$\text{Area} = \frac{h}{2}(b_1 + b_2) = 4(26) = \boxed{104}$.

5)



Using the two shaded △ and their midlines (aka bimedians) the length of the square's diagonal is 6, so its area $= \frac{d^2}{2} = \frac{6^2}{2} = \boxed{18}$.

6) $\sqrt{5x} - \sqrt{2x} = 5 - 2 \Rightarrow \sqrt{x}(\sqrt{5} - \sqrt{2}) = 5 - 2 = (\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$

Dividing, $\sqrt{x} = \sqrt{5} + \sqrt{2}$, so
$x = 5 + 2\sqrt{10} + 2 = \boxed{7 + 2\sqrt{10}}$.